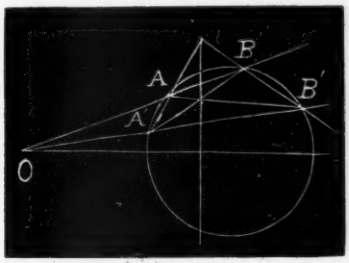
lines. OP is the polar of R with respect to the same lines; in other words the triangle OPR is self-conjugate.

To draw the polar of a point with respect to a conic: Draw any two secants through O, join AB', $A'B$, AA', BB'. The polar with respect to the conic is the same as the polar with respect to AB' and $A'B$, or with respect to AA' BB'.



Construction of Pole and Polar when center is given: The tangents at points (x_1, y_1) (x_2, y_2) of the curve are

$$axx_1+h(xy_1+x_1y)+byy_1+ g(x+x_1)+f(y+y_1)+c=0,$$

$$axx_2+h(xy_2+x_2y)+byy_2+g(x+x_2)+f(y+y_2)+c=0.$$

Subtraction gives

$$ax(x_1-x_2)+hx(y_1-y_2)+hy(x_1-x_2)+by(y_1-y_2)+g(x_1-x_2)+f(y_1-y_2)=0,$$

$$\text{or } (ax+hy+g)(x_1-x_2)+(hx+by+f)(y_1-y_2)=0,$$

the equation of a line through the intersection of the tangents. This line passes through the center, since the coördinates of the center cause $ax+hy+g$, $hx+by+f$, to vanish.

Putting $y_1-y_2=m(x_1-x_2)$, we may write this equation

$$(a+hm)x+(h+bm)y+(g+fm)=0.$$

This is the equation of the diameter which bisects the chord.

This may also be shown by putting $\frac{x_1+x_2}{2}$ for x, $\frac{y_1+y_2}{2}$ for y, the equation being satisfied thereby.

Finally: *The pole of a given line lies on the diameter which bisects the chord intercepted on that line by the conic.*

To find the pole of a given line: Draw the diameter conjugate to the line, join its extremities with any point of the line produced to cut the conic. Through the points of intersection draw a line to cut the diameter.

The construction in case of the parabola is easily effected by remembering that the center is at infinity.

A CLASS-BOOK OF NON-EUCLIDEAN GEOMETRY.

By DR. GEORGE BRUCE HALSTED.

America has taken a step in advance of all the world. She now has a text-book, a manual for class use, in non-Euclidean geometry. It is a pleasure to point out the close connection of THE AMERICAN MATHEMATICAL MONTHLY with this note-worthy achievement.

In this book, *Non-Euclidean Geometry*, by Henry Parker Manning, Ph. D., Assistant Professor of Pure Mathematics in Brown University, Ginn & Co., 1901, Pp. v+95, the method of treatment that has been taken as the basis of the first chapter, and which consequently underlies the other three chapters of the book, is that of Saccheri, drawn directly and solely from the pages of THE AMERICAN MATHEMATICAL MONTHLY where the first translation of Saccheri began to be published in June, 1894.

THE MONTHLY, Vol. I, p. 188, gives Saccheri's first proposition literally: "If two equal straights [sects] AC, BD, make with the straight AB angles equal toward the same parts: I say the angles at the join CD will be mutually equal." On page 189 is "Proposition II. The quadrilateral $ABCD$ remaining the same, the sides AB, CD are bisected in points M, and H. I say the angles at the join MH will be on both sides right."

Professor Manning paraphrases these two together on page 5: "If two equal lines in a plane are erected perpendicular to a given line, the line joining their extremities makes equal angles with them and is bisected at right angles by a third perpendicular erected midway between them."

In some respects we prefer the phraseology of Saccheri to that of Manning. Like the French, neither possesses a word for a piece of a straight line, German strecke, English *sect*. There is a kind of unbounded entity such that one and only one is on two points. This may be called *the straight*. It appears as an element in projective geometry. But it is of the essence of metric geometry that two points shall also completely determine an entity bounded by them, *the sect*, with which the idea of precise individual magnitude or quantity may be connected by setting up a conventional system of measurement. Distance is the result of the comparison of two sects. The distance between two points is the length of their sect in terms of a standard sect, say the centimeter.

Both the accepted popular and the accepted mathematical definitions of distance make it always a number, as, *e. g.* Wentworth, 1899, page 8, §50. "Def. The *distance* between two points is *the length* of the straight line [sect] that joins them;" and again the Cayley-Klein definition: "The distance between two points is equal to a constant times the logarithm of the cross-ratio in which the line joining the two points is divided by the fundamental quadric." Saccheri calls the two equal sects of his first proposition *straights*. Manning goes farther off, calling them *lines*.

B. A. W. Russell in "An Essay on the Foundations of Geometry" uses

the word 'distance' as a confounding and confusing designation for a sect itself and also for the numerical measures of that sect, whether by superposition, ordinary ratio, indeterminate as depending on the choice of a unit, or by projective metrics, indeterminate as depending on the fixing of the three points to be taken as constant in the varying cross ratios, these cross ratios themselves to be defined as numbers by the method of von Staudt, without presupposing ordinary measurement.

The confusion which may thus be introdued just from lack of a word is powerfully shown by the illustrious Poincaré in a response to Mr. Russell in the *Revue de Métaphysique et de Morale*, 1900, pp. 73—86, entitled "Sur les principes de la Géométrie." The following four sentences are curious in showing his results and at the same time showing the lack in French which causes a borrowing from German: "Comme le fait remarquer M. Halsted dans une brochure récente (*Science*, N. S. Vol. X, No. 251), M. Russell a eu tort d'employer indifférement le mot distance pour désigner ce que les Allemands appelent 'Strecke,' et en même temps la mesure de cette 'Strecke.' Le nom de distance ne convient qu' à la mesure de cette 'Strecke,' et cette mesure ne peut être définie que par une convention.

Si M. Russell n' avait pas, comme le lui reproche M. Halsted (vide supra), employé le mot distance dans deux sens différents, il n' y aurait plus d' apparence de cercle vicieux. Où serait cette apparence si l' on avait écrit: La distance est le résultat de la comparison de deux 'Strecken'."

Again, in both propositions Saccheri speaks of the *join* of two points. Manning paraphrases it as "the line joining" the two points. In a note to Euclid I. 5, Todhunter says of the phrase "Join FC"; "Custom seems to allow this singular expression as an abbreviation for 'draw the straight line FC', or for 'join F to C by the straight line FC.'" In Saccheri the join AB means the sect terminated by A and B. In projective geometry the join AB means the unbounded straight on A and B.

Under the heading *Definitions*, Saccheri says: "Since (from our first) the straight joining the extremities of equal perpendiculars standing upon the same straight (which we will call base), makes equal angles with these perpendiculars, three hypotheses are to be distinguished according to the species of these angles. And the first, indeed, I will call hypothesis of right angle; the second, however, and the third I will call hypothesis of obtuse angle, and hypothesis of acute angle." This Manning paraphrases as follows, under the heading *The Three Hypotheses*: "The angles at the extremities of two equal perpendiculars are either right angles, acute angles, or obtuse angles, at least for restricted figures. We shall distinguish the three cases by speaking of them as the hypothesis of the right angle, the hypothesis of the acute angle, and the hypothesis of the obtuse angle, respectively."

Saccheri's Prop. III. is: "If two equal straights, AC, BD, stand perpendicular to any straight, AB: I say the join CD will be equal, or less, or greater than AB, according as the angles at CD are right, or obtuse, or acute."

This Manning paraphrases as follows: "The line joining the extremities of two equal perpendiculars is, at least for any restricted portion of the plane, equal to, greater than, or less than the line joining their feet in the three hypotheses respectively."

In the same way is paraphrased Saccheri's Prop. IV., the converse of III.

Saccheri's Corollary about quadrilaterals with three right angles is given, page 12.

Saccheri's Prop. V. is: "The hypothesis of right angle, if even in a single case it is true, always in every case alone is true." In giving this, Manning writes: "If the hypothesis of *a* right angle," &c, evidently a slip for *the* right angle. Of course the Latin has no article.

Prop. VI. and Prop. VII. are combined, p. 13. Prop. IX. is reproduced on p 14. Prop. X. is given on p. 9.

In Prop. XI. Saccheri with the hypothesis of right angle demonstrates the celebrated Postulatum of Euclid, thus showing that his hypothesis of right angle is the ordinary Euclidean geometry. Manning does not reproduce this demonstration but says, p. 27: "The three hypotheses give rise to three systems of Geometry, which are called the Parabolic, the Hyperbolic, and the Elliptic Geometries. They are also called the Geometries of Euclid, of Lobachevski, and of Riemann." It should be noted that Manning's book gives only the simple elliptic, or single elliptic, or Clifford-Klein Geometry. It never even mentions the double elliptic or Spherical or Riemannian Geometry, which Killing maintains was the only form which ever came before Riemann's mind.

Manning's Chapter II., The Hyperbolic Geometry, seems taken bodily from Halsted's translation of Lobachevski's "Geometrical Researches on the Theory of Parallels." Though Halsted's translation of Bolyai is specifically mentioned on page 94, yet Manning shows no signs of having read it, and thus his book is confined within the bounds of propædeutics.

The most extraordinary two dozen pages in the history of thought is "The Science Absolute of Space," by Bolgai János. This is the most perfect case of genius. Take as example his §34: Through a given point to draw a parallel to a given straight. So simple; and yet neither Lobachevski nor anyone else ever reached it. It seems supernatural, uncanny. it makes one's hair stand on end. Perhaps he was, as he called himself, the phœnix of Euclid.

Manning's Chapter III., The Elliptic Geometry, pp. 62-8, is very brief, but what there is of it is good. The final Chapter IV., Analytic Non-Euclidean Geometry, is devoted to putting into coördinate and equational notation the new matters reached synthetically in the preceding chapters. The book ends with a Historical Note, pp. 91-5. This, in the main sound, may be in parts misleading. Thus it says, p. 91; "Legendre proved that the sum of the angles of a triangle can never exceed two right angles, and that if there is a single triangle in which this sum is equal to two right angles, the same is true of all triangles. This was, of course, on the supposition that a line is of infinite length." Now this beautiful theorem, if a single, then all triangles, Manning has in his own

book, p. 12, under the form "If the hypothesis of [a] right angle is true in a single case, it holds true in every case." proved by Saccheri without any supposition on the length of the line and a century before Legendre.

Again, not even the name of Schweikart is mentioned, though as I have shown in THE AMERICAN MATHEMATICAL MONTHLY, Vol. VII., pp. 247-252, and in "Science," Vol. XII., pp. 842-846, Schweikart may be considered the first to publish a genuine conscious treatise on Non-Euclidean Geometry [which I there give for the first time in English]. This fixes the date of the first conscious creation and naming of the Non-Euclidean Geometry as between 1812 and 1816.

We may perhaps timidly hope that the great Jesuit, Saccheri, had some suspicion of what he had really done. But meanwhile it seems almost certain that he really believed that his beautiful Non-Euclidean Geometry was all a reductio-ad-absurdum, and that he had really justified the title of his book, "Euclid Vindicated from Every Fleck," by proving Non-Euclidean Geometry untenable. On the other hand we of the new school, followers of Schweikart and Bolgai János, believe that it is our Non-Euclidean Geometry itself which finally vindicates Euclid from every fleck, and justifies the weighty tribute of Professor Alfred Baker: "Of the perfection of Euclid (B. C. 290) as a scientific treatise, of the marvel that such a work could have been produced two thousand years ago, I shall not here delay to speak. I content myself with making the claim that, as a historical study, Euclid is, perhaps, the most valuable of those that are taken up in our educational institutions."

Austin, Texas.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

115. Proposed by ALOIS F. KOVARIK, Instructor in Mathematics and Physics, Decorah Institute, Decorah Iowa.

Find the conditions of the coefficients of a general biquadratic equation so that it may be solved by quadratics.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa., and the PROPOSER.

To solve as a pure quadratic, without using the cubic, we might proceed as follows:

Let $x^4+ax^3+bx^2+cx+d=0$ be the general biquadratic.

(1) Then $(x^2+\frac{1}{2}ax)^2+bx^2-\frac{1}{4}a^2x^2+cx+d=0$.

Now if $c=\frac{1}{2}ab-\frac{1}{8}a^3$, we get $(x^2+\frac{1}{2}ax)^2+(b-\frac{1}{4}a^2)(x^2+\frac{1}{2}ax)+d=0$.

This equation can be solved by quadratics.

(2) Let $d=c^2/a^2$. Then $x^4+ax^3+bx^2+cx+c^2/a^2=0$.

$\therefore (x^2+\frac{p}{2}x+\frac{c}{a})^2=(\frac{p^2}{4}+\frac{2c}{a}-b)x^2$.

This is easily solved by quadratics.

This gives two conditions: 1st, $c=\frac{1}{2}ab-\frac{1}{8}a^3$; $d=c^2/a^2$.

II. Solution by **HARRY S. VANDIVER**, Bala, Pa.

The problem, I take it, is equivalent to the following:

What is the condition that the general biquadratic equation has a root which may be expressed in quadratic surds?

(A) Suppose the equation to be reduced to the form

$$x^4+qx^2+rx+s=0\ldots.(1),$$

where q, r, and s are rational; and assume that this is equivalent to

$$(x^2-kx+m)(x^2+kx+l)=0\ldots.(2);$$

then, following Descartes' method, we find as the cubic resolvent in k^2,

$$k^6+2qk^4+(q^2-4s)k^2-r^2=0\ldots.(3).$$

(*Hall and Knight's Higher Algebra*, page 485).

Now since x is to be found in terms of k, l, and m from (2), and, if it is to be in the form of a quadratic surd, then (3) must have a root in the form of a quadratic surd. Therefore, the roots of (3) must be of the forms $\pm\sqrt{b}$, and $\pm\sqrt{(c\pm\sqrt{d})}$, where b, c, and d are rational. It is sufficient to consider one value of k; put $k=\sqrt{b}$, and substituting in (2), we obtain results of the form

$$x=\begin{cases}\sqrt{m}+\sqrt{(g+n\sqrt{m})}\ldots.(4),\\ \sqrt{m}-\sqrt{(g+n\sqrt{m})}\ldots.(5),\\ -\sqrt{m}+\sqrt{(g-n\sqrt{m})}\ldots.(6),\\ -\sqrt{m}-\sqrt{(g-n\sqrt{m})}\ldots.(7),\end{cases}$$

where m, g, and n are rational.

These values may be taken as the general forms of quadratic surd roots of biquadratics. Hence the *condition that the general biquadratic has roots of the form* (4), (5), (6), and (7) *is that* (3) *has a root in the form* $\sqrt{b}$, *where* b *is rational.*

(B) *Special forms of biquadratics. Relations existing between the coefficients.*

Let the general biquadratic be

$$x^4+px^3+qx^2+rx+s=0\ldots.(9),$$

and assume that it can be thrown into the form

$$(ax^2+bx)^2+c(ax^2+bx)+d=0\ldots\ (10);$$

then, expanding, and removing the term involving x^3 (by the proper linear transformation) the coefficient of x also vanishes, and from this property we find that the relation between the coefficients of (9) is :

$$p^3-4pq+8r=0,$$

and in (10), the sum of two of the roots is equal to the sum of the other two.

(Cf. Burnside and Panton's *Theory of Equations*, page 41).

2nd Case. Assume that (9) can be put in the form

$$(mx+\frac{n}{x})^2+g(mx+\frac{n}{x})+f=0\ldots.(11).$$

Multiplying out, and comparing with (9), we find that the relation between the coefficients is $r^2=p^2s$, *and the roots of* (11) *are in geometrical progression.*

Also solved by *CHARLES C. CROSS, B. L. REMICK, J. SCHEFFER, H. C. WHITAKER,* and *J. W. YOUNG.*

116. Proposed by ARTEMAS MARTIN, A. M., Ph. D., LL. D., U. S. Coast and Geodetic Survey Office, Washington, D. C.

Solve the equations :

$$w(xy+yz+xz)=a\ ;\ x(wy+wz+yz)=b\ ;$$
$$y(wx+wz+xz)=c\ ;\ z(wx+wy+xy)=d.$$

I. Solution by G. B. M. ZEZR, A. M., Ph. D., The Temple College, Philadelphia, Pa.; J. M. HOWIE, The Nebraska State Normal School, Peru, Neb.; B. L. REMICK, Bradley Polytechnic Institute, Peoria, Ill.; NOAH ADAIR, A. M., Tyler College, Tyler, Tex.; G. I. HOPKINS, High School, Manchester, N.H.; FREMONT CRANE, B. S., C. E., Stockett, Mont.; H. C. WHITAKER, Ph.D., Manual Training School, Philadelphia, Pa., C. C. CROSS, Meredithville, Va.; L. B. FILLMAN, Chester, Pa.; O. S. WESTCOTT, A. M., Sc. D., North Division High School, Chicago, Ill.; and W. W. LANDIS, A. M., Dickinson College, Carlisle, Pa.

(1)+(2)+(3)+(4) gives $wxy+wxz+wyz+xyz=\frac{1}{3}(a+b+c+d)\ldots.(5)$.

(5)−(1) gives $xyz=\frac{1}{3}(b+c+d-2a)\ldots.(6)$.

(5)−(2) gives $wyz=\frac{1}{3}(a+c+d-2b)\ldots.(7)$.

(5)−(3) gives $wxz=\frac{1}{3}(a+b+d-2c)\ldots.(8)$.

(5)−(4) gives $wxy=\frac{1}{3}(a+b+c-2d)\ldots.(9)$.

(6)÷(7) gives $x/w=(b+c+d-2a)/(a+c+d-2b)$.

(6)÷(8) gives $y/w=(b+c+d-2a)/(a+b+d-2c)$.

(6)÷(9) gives $z/w=(b+c+d-2a)/(a+b+c-2d)$.

These values of x, y, and z in (1) give

$$w=\sqrt[3]{\frac{(a+b+c-2d)(a+b+d-2c)(a+c+d-2b)}{3(b+c+d-2a)^2}}=\sqrt[3]{\frac{BCD}{3A^2}},\ \text{suppose.}$$

Similarly, $x=\sqrt[3]{(ACD/3B^2)}$, $y=\sqrt[3]{(ABD/3C^2)}$, $z=\sqrt[3]{(ABC/3D^2)}$.

II. Soluiion by J. SCHEFFER, A. M., Hagerstown, Md.; M. A. GRUBER, A. M., War Department, Washington, D. C.; J. H. DRUMMOND, LL. D., Portland, Me.; ELMER SCHUYLER, B. Sc., Boys' High School, Reading, Pa.; COOPER D. SCHMITT, M. A., University of Tennessee, Knoxville, Tenn.; and H. S. VANDIVER, Bala, Pa.

Putting $xyzw=P$, the given equations change into the following:

$$P/z+P/y+P/x=a,\ P/z+P/y+P/w=b,$$
$$P/z+P/x+P/w=c,\ P/y+P/x+P/w=d.$$

Adding, we get $P/x+P/y+P/z+P/w=\frac{a+b+c+d}{3}$.

Putting $\frac{a+b+c+d}{3}=s$, and subtracting each of the above equations, gives

$$P/w=s-a,\ P/x=s-b,\ P/y=s-c,\ P/z=s-d.$$

Multiplying, $P^3=(s-a)(s-b)(s-c)(s-d)$.

$$\therefore x=\sqrt[3]{\frac{(s-a)(s-c)(s-d)}{(s-b)^2}},\ y=\sqrt[3]{\frac{(s-a)(s-b)(s-d)}{(s-c)^2}},$$

$$z=\sqrt[3]{\frac{(s-a)(s-b)(s-c)}{(s-d)^2}},\ w=\sqrt[3]{\frac{(s-b)(s-c)(s-d)}{s-a)^2}}.$$

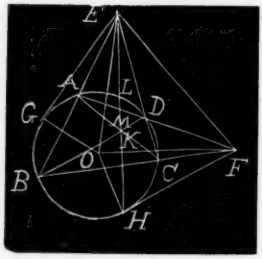
GEOMETRY.

146. Proposed by H. R. HIGLEY, M. Sc., Professor of Mathematics, Normal School, East Stroudsburg, Pa.

If the opposite sides of a quadrilateral inscribed in a circle be produced to meet, the square on the line joining the points of concurrence=the sum of the squares on the two tangents from these points. Ex. 24, page 219, Mackay's *Elements of Euclid*.

Solution by G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa., and J. SCHEFFER, A. M., Hagerstown, Md.

Let $ABCD$ be the inscribed quadrilateral; E, F the intersection of the opposite sides produced; O the center of the circle; M the intersection of the diagonals; EG tangent from E; FH, FL tangents from F.



Draw EF, EM and let EM cut FO in K. It has been shown that EM is the polar of F.

$\therefore$ EM passes through L, H and is perpendicular to OF.

$\therefore$ $EF^2=FK^2+FK^2$.

But $EK^2=EO^2-OK^2$, $FK^2=(FO-OK)^2$.

$\therefore$ $FK^2=FO^2-2FO.OK+OK^2$.

But $2FO.OK=2OH^2=2r^2$. $\therefore$ $EK^2+FK^2=EO^2+FO^2-2r^2$.

$EG^2=EO^2-r^2$, $FH^2=FO^2-r^2$.

$\therefore$ $EG^2+FH^2=EO^2+FO^2-2r^2=EK^2+FK^2=EF^2$.

147. Proposed by R. A. WELLS, Professor of Mathematics, Franklin College, Athens, Ohio.

Find the locus in space of the point which is equally illuminated by each of two unequal lights whose intensities are a and b $(a>b)$, placed at a distance c from each other.

Solution by WILLIAM HOOVER, A.M., Ph.D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

Let (x, y, z) be the rectangular coördinates of any one of the illuminated points, the mid-point of the line joining the two lights taken along the x-axis as origin; then the coördinates of the two lights may be given as $(l, 0, 0)$, $(-l, 0, 0)$, and by theory

$$\frac{a}{(x-l)^2+y^2+z^2}=\frac{b}{(x+l)^2+y^2+z^2}\ldots.(1),$$

or $x^2+y^2+z^2+2l\frac{a+b}{a-b}x+l^2=0\ldots.(2)$, a sphere.

Also solved by *G. B. M. ZERR*, and *J. SCHEFFER*.

148. Proposed by DR. E. D. ROE, JR., Associate Professor of Mathematics in Syracuse University, Syracuse, N. Y.

The condition that two triangles, abc, xyz, are similar is

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ x & y & z \end{vmatrix}=0,$$

and the condition that the triangle abc is equilateral is

$$\begin{vmatrix} a & b & 1 \\ b & c & 1 \\ c & a & 1 \end{vmatrix}=0.$$

(Used in solving 130.)

I. Solution by the PROPOSER.

First. *The given conditions are necessary*, for if two triangles abc, xyz, be similar, then $\frac{a-b}{x-y}=\frac{b-c}{y-z}=\frac{c-a}{z-x}=r\ldots.(1)$, for this expresses not only that the homologous sides are proportional, but also that the homologous angles are equal. (1) is equivalent to the equations

$$\begin{aligned} a-b&=r(x-y)\\ b-c&=r(y-z)\\ c-a&=r(z-x) \end{aligned}\quad\ldots(2).$$

Multiplying the equations (2) by x, y, and z, respectively, and adding, we get

$$\Sigma(a-b)z=r\Sigma(x-y)z=0\ldots.(3),$$

or $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ x & y & z \end{vmatrix}=0\ldots.(4)$, which is therefore necessary at least, if the triangles are similar.

If the triangle abc is equilateral, it is also equiangular, and

$$\frac{a-b}{b-c}=\frac{b-c}{c-a}=\frac{c-a}{a-b}=r\ldots.(5), \text{ or}$$

$$\begin{aligned} a-b&=r(b-c)\\ b-c&=r(c-a)\\ c-a&=r(a-b) \end{aligned} \quad \ldots.(6).$$

Multiplying these equations by a, b, c, respectively, and adding, we have

$$\Sigma a^2-\Sigma ab=r\Sigma(b-c)a=0\ldots.(7), \text{ or}$$

$$\begin{vmatrix} a & b & 1\\ b & c & 1\\ c & a & 1 \end{vmatrix}=0\ldots.(8),$$

and this condition is also at least necessary.

Second. *These conditions are also sufficient.* For, with

$$\frac{a-b}{x-y}=r_1,\quad \frac{b-c}{y-z}=r_2,\quad \frac{c-a}{z-x}=r_3\ldots.(9),$$

or what is the same thing, with

$$\begin{aligned} a-b&=r_1(x-y)\\ b-c&=r_2(y-z)\\ c-a&=r_3(z-x) \end{aligned} \quad \ldots.(10)$$

$$\text{form}\begin{vmatrix} 1 & 1 & 1\\ a & b & c\\ x & y & z \end{vmatrix}=(a-b)z+(b-c)x+(c-a)y$$

$$=r_1z(x-y)+r_2x(y-z)+r_3y(z-x)\ldots.(11),$$

and put this equal to zero. We are then to prove in fact that this obliges r_1, r_2, r_3 to be equal to one another. We have

$$r_1z(x-y)+r_2x(y-z)+r_3y(z-x)=0\ldots.(12).$$
$$r_1(x-y)+r_2(y-z)+r_3(z-x)=0\ldots.(13).$$

The latter equation comes by adding equations (10) together. By solving (12) and (13),

$$\frac{r_1}{r_3}=\frac{\begin{vmatrix} x(y-z) & y(z-x) \\ (y-z) & (z-x) \end{vmatrix}}{\begin{vmatrix} z(x-y) & x(y-z) \\ (x-y) & (y-z) \end{vmatrix}}=\frac{(z-x)\begin{vmatrix} x & y \\ 1 & 1 \end{vmatrix}}{(x-y)\begin{vmatrix} z & x \\ 1 & 1 \end{vmatrix}}=1.$$

$$\frac{r_2}{r_3}=\frac{\begin{vmatrix} y(z-x) & z(x-y) \\ (z-x) & (x-y) \end{vmatrix}}{\begin{vmatrix} z(x-y) & x(y-z) \\ (x-y) & (y-z) \end{vmatrix}}=\frac{(z-x)\begin{vmatrix} y & z \\ 1 & 1 \end{vmatrix}}{(y-z)\begin{vmatrix} z & x \\ 1 & 1 \end{vmatrix}}=1. \quad \ldots.(14).$$

or $r_1=r_3$, $r_2=r_3$, and therefore $r_1=r_2=r_3$ . . .(15), whence the homologous sides of the triangles abc, xyz are proportional, and their homologous angles are equal, and the triangles are in fact similar. In the other case form

$$\begin{vmatrix} a & b & 1 \\ b & c & 1 \\ c & a & 1 \end{vmatrix}=r_1a(b-c)+r_2b(-a)+r_3c(a-b)$$

$$\text{where } r_1=\frac{a-b}{b-c},\quad r_2=\frac{b-c}{c-a},\quad r_3=\frac{c-a}{a-b} \ldots.(16),$$

and put this equal to zero. We have

$$\begin{aligned} &r_1a(b-c)+r_2b(c-a)+r_3c(a-b)=0. \\ &r_1(b-c)+r_2(c-a)+r_3(a-b)=0. \quad \ldots.(17). \\ &r_1r_2r_3=1. \end{aligned}$$

The last two of equations (17) come from (16). As before, by solving the first two equations of (17) for r_1/r_3, r_2/r_3, we get $r_1=r_2=r_3=r$ say; by the last of equation (17) $r^3=1$, therefore

$$\begin{aligned} &r=1 \\ &r=\cos 60^\circ+i\sin 60^\circ \quad \ldots.(18), \\ &r=\cos 120^\circ+i\sin 120^\circ \end{aligned}$$

and the only solution applying to a geometrical triangle in the plane is the second, which in fact determines an equilateral triangle, since modulus $r=1$, argument $r=60^\circ$.

Since the conditions are both necessary and sufficient, they are the required conditions.

II. Solution by LON C. WALKER, Assistant in Mathematics, Leland Stanford Jr. University, Palo Alto, Cal.

Let z_1, z_2, z_3 represent the vertices of a triangle in a z-complex-number-

plane, and let z'_1, z'_2, z'_3 represent the corresponding vertices in a z'-complex-number-plane.

A transformation can always be found that will change z_1 in z'_1, and z_2 into z'_2. For this it is sufficient that a and b in

$$\left.\begin{aligned} z'_1 &= az+b \\ z'_2 &= az_2+b \end{aligned}\right\} \ldots\ldots(1),$$

shall satisfy the condition

$$a=\frac{z'_2-z'_1}{z_2-z_1}, \text{ and } b=\frac{z'_1z_2-z'_2z_1}{z_2-z_1}.$$

Every transformation of the plane into itself that leaves every figure in the plane similar to itself is expressible in the form

$$z'=az+b \ldots (2).$$

The value of a and b in (1) must satisfy the equation $z'_3=az_3+b$.

For this the necessary and sufficient condition is

$$\frac{z'_3-z'_1}{z_3-z_1}=\frac{z'_2-z'_1}{z_2-z_1}; \quad \therefore \begin{vmatrix} z'_1, & z_1, & 1 \\ z'_2, & z_2, & 1 \\ z'_3, & z_3, & 1 \end{vmatrix} =0.$$

The conditions that the sides shall be equal are

$$\frac{z_1-z_2}{z_2-z_3}=\frac{z_2-z_3}{z_3-z_1}=\frac{z_3-z_1}{z_1-z_2}; \quad \therefore \begin{vmatrix} z_1, & z_2, & 1 \\ z_2, & z_3, & 1 \\ z_3, & z_1, & 1 \end{vmatrix} =0.$$

Also demonstrated by *WILLIAM HOOVER, G. B. M. ZERR, J. W. YOUNG*, and *J. SCHEFFER*.

CALCULUS.

108. Proposed by JOHN M. COLAW, A. M., Monterey, Va.

The hypotenuse of a plane right triangle increases uniformly at the rate of 1-12 of an inch a second. If the legs are as 2 to 3, at what rate is the area of the triangle increasing when the perpendicular from the right angle upon the hypotenuse is 12 inches?

Solution by C. HORNUNG, A. M., Heidelberg University, Tiffin, O.; H. C. WHITAKER, Ph. D., Manual Training School, Philadelphia, Pa.; J. W. YOUNG, Cornell University, Ithaca, N. Y.; P. S. BERG, Larimore, N. D.; and D. G. DORRANCE, Jr., Camden, N. Y.

Let x=the length of the hypotenuse, and y=the area.

Then $dx=\frac{1}{12}$, and dy is required when the perpendicular from the right angle to the hypotenuse equals 12.

From the condition of the problem $y=\frac{3x^2}{13}$, whence $dy=\frac{6xdx}{13}$.

Now when the perpendicular from the right angle to the hypotenuse is 12 the hypotenuse must be 26. Hence, substituting in the last equation we have $dy=\frac{6\times 26\times \frac{1}{12}}{13}=1$; *i. e.*, the area, at the time mentioned, is increasing at the rate of 1 square inch a second.

Also solved by *L. B. FILLMAN, ALOIS F. KOVARIK, J. SCHEFFER, C. D. SCHMITT*, and *G. B. M. ZERR*.

109. Proposed by M. E. GRABER, Heidelberg University, Tiffin, Ohio.

Find the curve in which the product of the perpendiculars drawn from two fixed points to any tangent is constant.

Solution by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn., and the PROPOSER.

Let the equation of the tangent be

$$y - y' = \frac{dy'}{dx'}(x-x').$$

And let the two points be $(a, 0)$ and $(-a, 0)$.

The product of the two perpendiculars is easily found to be

$$\frac{(xdy-ydx)^2-(ady)^2}{(dx)^2+(dy)^2} \text{ which} = b^2,$$

$$\text{or } \frac{(xp-y)^2-(ap)^2}{1+p^2}=b^2, \text{ or } (xp-y)^2=p^2(a^2+b^2)+b^2,$$

$$\text{or } y=px\pm\sqrt{[p^2(a^2+b^2)+b^2]}.$$

This is Clairaut's form, so that we have

$$y=mx\pm\sqrt{[m^2(a^2+b^2)+b^2]},$$

which is the well-known tangent to an ellipse.

Also solved by *H. C. WHITAKER*, and *G. B. M. ZERR*.

MECHANICS.

108. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pa.

Can it be shown, as a result of Kepler's third law, that the distances are inversely proportional to the squares of the velocities?

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

This can be demonstrated for circular orbits as follows:

$t^2 : t_1^2 = a^3 : a_1^3$; but $t^2 = 4\pi^2 a^2/v^2$, $t_1^2 = 4\pi^2 a_1^2/v_1^2$.

$\therefore\ 1/v^2 : 1/v_1^2 = a : a_1$.

For elliptic orbits

$$v^2 = \frac{\mu}{b}\left(\frac{2b-r}{r}\right);\quad v_1^2 = \frac{\mu}{b_1}\left(\frac{2b_1 - R}{R}\right)$$

where b, b_1 are the semi-major axes, respectively.

$$\therefore \frac{1}{v^2} : \frac{1}{v_1^2} = \frac{br}{2b-r} : \frac{b_1 R}{2b_1 - R} = r : R \text{ when } r = R = \infty.$$

$\therefore$ The theorem is true for parabolic orbits.

109. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pa.

If the sun were moved into the center of the earth's orbit, how much would the present length of the year be changed?

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

It is presumed in this solution that the earth's orbit remains the same.

Let t=the present length=1 year; T=the required length; μ=absolute force; and a=semi-major axis.

$$\therefore\ t = 2\pi\sqrt{\frac{a^3}{\mu}},\quad T = \frac{2\pi}{1/\mu}.\qquad \therefore\ \frac{t}{T} = a_1/a \text{ or } T = \frac{t}{a_1/a}.$$

But $a=1$, and $t=1$ year.

$\therefore\ T=1$ year and the length of the year would remain the same.

110. Proposed by W. H. DRANE, Graduate Student, Harvard University, Cambridge, Mass.

A tight roll of very thin and perfectly flexible oilcloth is placed upon a rough inclined plane, a portion of the cloth being unrolled, and, extending from underneath the roll, is spread out smoothly upon the inclined plane below. The roll is then allowed to descend under the action of gravity, picking up the cloth as it goes. Determine the motion as far as possible.

No solution of this problem has been received.

111. Proposed by WALTER H. DRANE, Graduate Student, Harvard University, Cambridge, Mass.

A man, weighing 150 pounds at the surface of the earth, ascends in a balloon until the area visible to him is $2\pi R^2(1-\frac{1}{2}\sqrt{2})$. What is his weight at that height?

Solution by C. HORNUNG, A. M., Heidelberg University, Tiffin, O.; P. S. BERG, B. Sc., Larimore, N. D.; J. SCHEFFER, A. M., Hagerstown, Md.; G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa.

If R=the radius of the earth, and h=the altitude of the zone visible, then

$$2\pi Rh = 2\pi R^2(1-\tfrac{1}{2}\sqrt{2}),\text{ whence } h = R(1-\tfrac{1}{2}\sqrt{2}).$$

Now if H be the distance of the balloonist from the earth's center, then

$$H:R::R:R-h \text{ or } H=R_1/2.$$

Now by the law of gravitation, $(R_1/2)^2:R^2::150:x$, the required weight, that is, $x=\dfrac{150R^2}{2R^2}=75$.

112. Proposed by F. P. MATZ. M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pa.

Vary the radius of curvature of a plane curve inversely as the abscissa; then the solution will give you, (1) Ryan's Equation of the Elastic Curve, and (2) Wood's Equation of the Hydrostatic Curve.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

$$\text{Radius of curvature}=\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}=\frac{a^2}{2x}.$$

$\therefore a^2(d^2y/dx^2)=2x[1+(dy/dx)^2]^{\frac{3}{2}}$. Let $dy/dx=p$.

$\therefore a^2(dp/dx)=2x(1+p^2)^{\frac{3}{2}}$ or $a^2p/\sqrt{(1+p^2)}=(x^2+A)$.

$\therefore p=dy/dx=\pm\sqrt{(x^2+A)}/\sqrt{[a^4-(x^2+A)^2]}$. Let $x^2+A=a^2\cos\theta$.

$$\therefore y=\mp\tfrac{1}{2}a^2\int\frac{\cos\theta d\theta}{\sqrt{(a^2\cos\theta-A)}}=\mp\tfrac{1}{2}a^2\int\frac{(1-2\sin^2\frac{1}{2}\theta)d\theta}{\sqrt{(a^2+1-A-2\sin^2\frac{1}{2}\theta)}}.$$

Let $\frac{1}{2}\theta=\varphi$ and $2/(a^2+1-A)=e^2$.

$$\therefore y=\pm\frac{a^2e^2}{2}\int\frac{(1-2\sin^2\varphi)d\varphi}{\sqrt{(1-e^2\sin^2\varphi)}}.$$

$\therefore y=\pm a^2E(e,\ \varphi)\mp(\frac{1}{2}a^2)(2-e^2)F(e,\ \varphi)+B\ldots.(1)$.

Since $\varphi=\frac{1}{2}\theta$, $y=\pm a^2E(e,\ \frac{1}{2}\theta)\mp(\frac{1}{2}a^2)(2-e^2)F(e,\ \frac{1}{2}\theta)+B\ldots.(2)$.

(1) represents Ryan's Elastic Curve and (2) represents the Hydrostatic Curve. In the above $e<1$ and $A<a^2+1$.

In the second equation the curve can never cross the line of force since $\sin(\frac{1}{2}\theta)$ cannot equal $1/e$.

Also solved by *J. SCHEFFER*.

DIOPHANTINE ANALYSIS.

58. Proposed by JOHN M. COLAW, A. M., Monterey, Va.

Find a square proper fraction which if subtracted from unity will leave for remainder a square proper fraction.

I. Solution by C. E. ARMENTROUT, Professor of Mathematics, Rockingham Military Institute, Mt. Crawford, Va.; G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa.; and P. S. BERG, Larimore, N. D.

Let $(y/x)^2$ be the square proper fraction.

$1-(y/x)^2=(n/m)^2$ or $m^2(x^2-y^2)=n^2x^2$.

Let $x=p^2+q^2$, $y=p^2-q^2$.

$\therefore\ 1-\left(\frac{p^2-q^2}{p^2+q^2}\right)=\frac{4p^2q^2}{(p^2+q^2)^2}$, where p can be any value greater than q.

Let $p=2$, $q=1$. $\therefore\ 1-(\frac{3}{5})^2=(\frac{4}{5})^2$.

Let $p=3$, $q=2$. $\therefore\ 1-(\frac{5}{13})^2=(\frac{12}{13})^2$.

II. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C., and COOPER D. SCHMITT, A. M., University of Tennessee, Knoxville, Tenn.

This reduces to the simple statement that $a^2/c^2+b^2/c^2=1$, or that $a^2+b^2=c^2$. This is evidently satisfied in the case of every rational right triangle.

Thus, $3^2+4^2=5^2$, or $\frac{9}{25}+\frac{16}{25}=1$.

This is the same as saying $1-\frac{9}{25}=\frac{16}{25}$, thus fulfilling the condition of the problem.

Similarly an indefinite number of examples might be given.

Also solved by *J. H. DRUMMOND, ALOIS F. KOVARIK, J. W. YOUNG*, and the late *SYLVESTER ROBINS*.

86. Proposed by A. H. BELL, Hillsboro, Ill.

The edges of a rectangular parallelopiped are within one of the proportion 3:6:7, and if they are $3x$, $6x\mp1$, $7x$, then $(3x)^2+(6x\mp1)^2+(7x)^2$=the diagonal squared$=94x^2\mp12x+1=\square$. To find four integral values for x in this equation.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let $94x^2\mp12x+1=(mx\mp1)^2=m^2x^2\mp2mx+1$.

$\therefore\ x=\pm\frac{2(m-6)}{94-m^2}$.

Expanding $\sqrt{94}$ as a continued fraction we get

$$\sqrt{94}=9+\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{5+}\,\frac{1}{1+}\,\frac{1}{8+}\,\frac{1}{1+}\,\frac{1}{5+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{9+}\,.$$

The successive convergents are as follows:

$$\frac{9}{1},\ \frac{10}{1},\ \frac{29}{3},\ \frac{97}{10},\ \frac{126}{13},\ \frac{223}{23},\ \frac{1241}{128},\ \frac{1644}{151},\ \frac{12953}{1336},\ \frac{14417}{1487},$$

$$\frac{85038}{8771},\ \frac{99455}{10258},\ \frac{184493}{19029},\ \frac{652934}{67345},\ \frac{1490361}{1537191},\ \frac{2143295}{221064},\ \cdots\ \cdots$$

The first integral value of x is given by substituting for m the eighth convergent, the second by using the sixteenth convergent for m.

Let $m=\frac{1464}{151}$, $x=84258$, $3x=252774$, $6x-1=505547$, $7x=589806$, $94x^2-12x+1=(816911)^2$.

Let $m=-\frac{1464}{151}$, $x=357870$. $3x=1073610$, $6x+1=2147221$, $7x=2505090$, $94x^2+12x+1=(3469679)^2$.

Let $m=\frac{2143295}{2210644}$, then $x=361179226608$, $3x=1083537679824$, $6x-1=2167075359647$, $7x=2528254586256$, $94x^2-12x+1=(3501762523489)^2$.

Let $m=-\frac{2143295}{2210644}$, then $x=1534042236912$, $3x=4602126710736$, $6x+1=9204253421471$, $7x=10738295658384$, $94x^2+12x+1=(14873091304609)^2$.

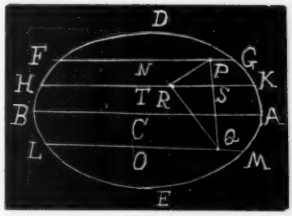
AVERAGE AND PROBABILITY.

95. Proposed by G. B. M. ZERR. A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Three random points are taken in an ellipse, one on each side of the major axis, and the third anywhere in the ellipse. Find the average area of the triangle formed by joining the three points.

Solution by the PROPOSER.

Let $ADBE$ be the ellipse, AB the major and DE the minor axis, C the center, and PQR the triangle formed by joining the random points. Through P, Q, R draw the chords FG, LM, HK parallel to AB.



It is only necessary to consider the relative positions in which the chord HK lies between the chords FG and LM.

Let $CD=CE=b$, $CA=CB=a$, $CN=u$, $CO=v$, $CT=w$, $NP=x$, $OQ=y$, $TR=z$, $TS=t$, $NG=x'$, $OM=y'$, $TK=z'$.

Then $x'=(a/b)\sqrt{(b^2-u^2)}$, $y'=(a/b)\sqrt{(b^2-v^2)}$, $z'=(a/b)\sqrt{(b^2-w^2)}$, $t=y-\frac{(y-x)(w-v)}{u-v}$.

Area $PQR=\frac{1}{2}(t-z)(u-v)=A$, $t>z$. Area $PQR=\frac{1}{2}(z-t)(u-v)=A_1$, $t<z$.

The limits of u are 0 and b; of v, $-b$ and 0; of w, v and u; of x, $-x'$ and x'; of y, $-y'$ and y'; of z, $-z'$ and t, and t and z'.

Hence, since the whole number of ways the three points can be taken is $\frac{1}{4}\pi^3a^3b^3$ and doubling since P and Q are interchangeable the required average area is

$$\triangle=\frac{8}{\pi^3a^3b^3}\int_0^b\int_{-x'}^{x'}\int_{-b}^{0}\int_{-y'}^{y'}\int_v^u\left[\int_{-z'}^{t}A\,dz+\int_t^{z'}A_1\,dz\right]du\,dx\,dv\,dy\,dw$$

$$= \frac{4}{\pi^3 a^3 b^3} \int_0^b \int_{-x}^{x'} \int_{-b}^{0} \int_{-y}^{y'} \int_v^u \left[\frac{a^2}{b^2}(b^2 - w^2) + \left(y - \frac{(y-x)(w-v)}{u-v} \right)^2 \right]$$

$$(u-v)\,du\,dx\,dv\,dy\,dw$$

$$= \frac{4}{3\pi^3 a^3 b^3} \int_0^b \int_{-x}^{x'} \int_{-b}^{0} \int_{-y}^{y'} [3a^2 - (a^2/b^2)(u^2 + uv + v^2) + y^2 + xy + x^2]$$

$$(u-v)^2\,du\,dx\,dv\,dy$$

$$= \frac{4}{9\pi^3 a^2 b^4} \int_0^b \int_{-x}^{x'} \int_{-b}^{0} \left[20a^2 + 6x^2 - \frac{6a^2u^2}{b^2} - \frac{6a^2uv}{b^2} - \frac{8a^2v^2}{b^2} \right]$$

$$\sqrt{(b^2 - v^2)}\,(u-v)^2\,du\,dx\,dv$$

$$= \frac{2}{36\pi^3 a^2 b^4} \int_0^b \int_{-x'}^{x'} (39\pi a^2 b^2 u^2 + 8\pi a^2 b^4 + 12\pi b^2 u^2 x^2 - 12\pi a^2 u^4 + 3\pi b^4 x^2$$

$$+ 96a^2 b^3 u - 16a^2 bu^3 + 32b^3 u x^2)\,du\,dx$$

$$= \frac{2a}{54\pi^3 b^5} \int_0^b (126\pi b^2 u^2 - 48\pi u^4 + 27\pi b^4 + 320b^3 u - 80bu^3) \sqrt{(b^2 - u^2)}\,du$$

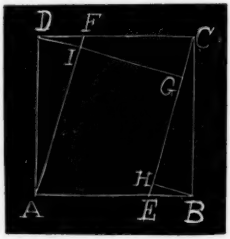
$$= \left(\frac{35}{72\pi} + \frac{32}{9\pi^3} \right) ab.$$

96. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics in Drury College, Springfield, Mo.

A random straight line is drawn across a square; find the chance that it intersects two opposite sides. [From *Byerly's Integral Calculus*, page 209].

Solution by the PROPOSER.

Let $ABCD$ be the given square whose side is a. Draw the parallel lines AF and CE. Then all lines parallel to AF and lying between AF and CE will intersect two opposite sides of the square. The total number of such lines is proportional to IG. The total number of lines parallel to AF and crossing the square is proportional to $DG + BH$.



Hence, the probability of a line drawn parallel to AF and intersecting the opposite sides of the square is

$$p_0 = \frac{IG}{DF + BH}.$$

Let the angle $FAB = \theta$. Then $DG = CD\sin\theta = a\sin\theta$, $IG = DG - DI = DC\sin\theta - AD\cos\theta = a(\sin\theta - \cos\theta)$, and $BH = EB\sin\theta = BC\cot\theta\sin\theta = a\cos\theta$.

$$\therefore p_0 = \frac{a(\sin\theta - \cos\theta)}{a(\sin\theta + \cos\theta)}.$$

$$\therefore\ p=\frac{a[(\sin\theta'-\cos\theta')+(\sin\theta''-\cos\theta'')+(\sin\theta'''-\cos\theta''')+\text{ etc.}]}{a[(\sin\theta'+\cos\theta')+(\sin\theta''+\cos\theta'')+(\sin\theta'''+\cos\theta''')+\text{ etc.}]}$$

$$=\frac{a\int(\sin\theta-\cos\theta)d\theta}{a\int(\sin\theta+\cos\theta)}.$$

The limits of θ for favorable cases are $\frac{1}{2}\pi$ and $\frac{1}{4}\pi$ and doubled, for the total cases $\frac{1}{2}\pi$ and 0.

$$\therefore\ p=\frac{2\int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi}[\sin\theta-\cos\theta]d\theta}{\int_{0}^{\frac{1}{2}\pi}[\sin\theta+\cos\theta]d\theta}=(\sqrt{2}-1).$$

NOTE.—The answer given in Professor Byerly's Integral Calculus, Edition of 1890, is $\frac{1}{4}-1/\pi\log 2$.

This is the answer to the following problem: *From a point taken at random in the side of a square, a line is drawn at random across the square. What is the chance that the line will intersect the opposite side of the square?*

Professor Zerr interpreted the problem in this way and solved it accordingly.

Solutions somewhat similar to the one above were received from J. M. Colaw and Lon C. Walker.

Professor Walker sent a solution of No. 93. His answer is $\frac{1}{2}\frac{1}{4}\pi p^2$. This difference in results is due to the different interpretations of the problem by Professors Zerr and Walker.

97. Proposed by L. C. WALKER, Assistant Professor of Mathematics, Leland Stanford Jr. University, Palo Alto, Cal.

A straight line is drawn at random across a circle, and five points are taken at random in the surface of the circle. Required the chance that all the points are on the same side of the line.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let θ=angle subtended by random chord at the center. Then area of the smaller segment is $a^2(\theta-\sin\theta\cos\theta)$. Hence the required chance is

$$p=\frac{\int_0^{\pi}2[a^2(\theta-\sin\theta\cos\theta)]^5 a\sin\theta d\theta}{\int_0^{\pi}(\pi a^2)^5 a\sin\theta d\theta}$$

$$=\frac{1}{\pi^5}\int_0^{\pi}(\theta-\sin\theta\cos\theta)^5\sin\theta d\theta=1-\frac{128}{9\pi^2}+\frac{2768896}{33075\pi^4}=.4184.$$

MISCELLANEOUS.

89. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Find the earth's average density and mass, having given that the attraction of a ball of lead 2 feet in diameter, on a particle placed close to its surface, is less than the earth's attraction in the ratio 1 : 10250000, and that the density of lead is $11\frac{1}{2}$ times that of water.

Solution by the PROPOSER.

Regard the earth as a sphere, radius 3956 miles.

$3956 \times 5280 = 20887680$ feet.

Let D be the earth's average density, then since the attractions at their surfaces are proportional to their densities multiplied by their radii, we get at once, $1 : 10250000 = 11.5 \times 1 : D \times 20887680$.

$\therefore D = 5.643$.

The mass of the earth from these data is $\frac{4}{3}\pi(20887680)^3 \times 5.643 \times 62.5$ pounds $= m$.

$\therefore m = 1477.3374(20887680)^3$ pounds $= 738.6687(2088768)^3$ tons.

Also solved by J. SCHEFFER.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

142. Proposed by M. J. CRAWFORD, Principal of Crawford's Academy, Savannah, Ga.

A gentleman has a garden 400 feet long and 300 feet wide, which he wishes to raise 9 inches higher by means of the earth to be dug out of a ditch 6 feet wide and surrounding the entire garden. How deep must the ditch be?

143. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pa.

A's income $= a/b$th part $= \frac{3}{4}$ of B's income. A's outgo $= m/n$th part $= \frac{1}{2}$ of B's income. B's outgo $= p/q$th part $= 1/1$ of A's income. What is the ratio of their savings?

*** Solutions of these problems should be sent to B. F. Finkel not later than June 10.

ALGEBRA.

134. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pa.

Solve neatly and briefly the equations

$x^3 + x^2y + y^3 = 53 \ldots (1)$, $y^3 + y^2z + z^3 = 13 \ldots (2)$, and $z^3 + z^2x + x^3 = 31 \ldots (3)$.

135. Proposed by CHARLES C. CROSS, Whaleyville, Va.

Tangents parallel to the three sides are drawn to the in-circle. If p, q, r, be the lengths of the parts of the tangents within the triangle, prove that

$$\frac{p}{a}+\frac{q}{b}+\frac{r}{c}=1.$$

*** Solutions of these problems should be sent to J. M. Colaw not later than June 10.

GEOMETRY.

163. Proposed by J. C. NAGLE, Professor of Civil Engineering in the Agricultural and Mechanical College of Texas, College Station, Texas.

Given the equal sides of an isosceles triangle and the radius of the inscribed circle to solve the triangle. As a numerical example let the known sides be 27 and the radius of the inscribed circle 3.5. The problem occured in connection with some mill work and the exterior angles of the triangle were required in order to make patterns for iron braces.

164. Proposed by J. M. HARCOURT, M. D., 305 Clinton Street, Brooklyn, N. Y.

Given two tangents to a parabola, find the locus of the center of the nine-point circle of the triangle by the two given tangents and any third tangent.

165. Proposed by W. H. ECHOLS, B.Sc., C.E., Professor of Mathematics, University of Virginia, Charlottsville, Va.

$OB=b$, $OA=a$ are the semi-conjugate diameters of an ellipse. Draw BM perpendicular to and equal to OA, cutting it in N. Show that as M slides on the fixed line OM and N on OA the point B traces the curve.

*** Solutions of these problems should be sent to B. F. Finkel not later than June 10.

CALCULUS.

128. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

The differential equation of a curve is $\frac{d^3y}{dx^3}+y=0$. Find its equation, there being the additional conditions that for $x=0$, $y=1$, that the tangent at the point (0, 1) makes an angle of 45° with the axes, and finally that that point is a point of inflexion.

129. Proposed by JOHN M. COLAW, A. M., Monterey, Va.

Among all quadrilaterals inscribed in an ellipse, to determine that which contains the greatest area.

*** Solutions of these problems should be sent to J. M. Colaw not later than June 10.

MECHANICS.

120. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, Gloucestershire, England.

I project an elastic particle along a chord c of a smooth fixed circular rim of diameter d. The coefficient of elasticity between the particle and the rim is e, and the particle continually rebounds. Find the length of the chord described after the nth rebound.

121. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, Gloucestershire, England.

Two equal scale pans of mass m hang at rest over a smooth pulley. An inelastic particle, mass M, is dropped from a height h into one pan, and simultaneously another of equal mass and elasticity e is dropped from the same height into the other. Prove that every impact occurs when the pans are in their original positions, and find the total space described by either pan before motion ceases.

*** Solutions of these problems should be sent to B. F. Finkel not later than June 10.

AVERAGE AND PROBABILITY.

104. Proposed by LON C. WALKER, Assistant in Mathematics, Leland Stanford Jr. University, Palo Alto, Cal.

In a given sphere two radii are drawn at random, and a point taken in each at random. (1) Find the chance that the distance between the two points does not exceed the radius of the sphere. (2) Find the distance between them.

105. Proposed by LON C. WALKER, Assistant in Mathematics, Leland Stanford Jr. University, Palo Alto, Cal.

Find the average distance of the center of an ellipsoid, axes $2a$, $2b$, and $2c$, from its surface.

106. Proposed by LON C. WALKER, Assistant in Mathematics, Leland Stanford Jr. University, Palo Alto, Cal.

Required the average distance between two points in a hemisphere.

*** Solutions of these problems should be sent to B. F. Finkel not later than June 10.

MISCELLANEOUS.

105. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

If the refractive index of a medium at any point be $\mu = x$, prove that the path of the ray will be the curve $\frac{2x}{a} = \frac{c}{a} l^{y/a} + \frac{a}{c} l^{-(y/a)}$, a and c being constants.

106. Proposed by J. W. YOUNG, Oliver Graduate Student in Mathematics, Cornell University, Ithaca, N. Y.

Prove that $\frac{(2m)!}{(m!)^2}$ is an integer; and more generally that $\frac{(nm)!}{(m!)^n}$ is an integer, m, n being any positive integers.

107. Proposed by WILLIAM HOOVER, A.M., Ph.D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

The index of refraction of a medium varying inversely as the square root of the distance, prove that the path of a ray of light in the medium is a cycloid.

*** Solutions of these problems should be sent to J. M. Colaw not later than June 10.

NOTES.

Dr. G. A. Miller, of Cornell University, has been appointed to an issistant professorship of mathematics at Leland Stanford Jr. University.

Miss R. H. Vivian, Alumnæ Fellow in Mathematics at the University of Pennsylvania, has been appointed Instructor in Mathematics at Wellesley.

Professor D. E. Smith, principal of the Brockport, N. Y., State Normal School, has been called to a professorship of mathematics in Teachers College, Columbia University.

Dr. J. M. Rice died at Northboro, Mass., on March 2d, aged sixty-eight years. He was appointed Professor of Mathematics at the Naval Academy at Annapolis in 1870. In 1879, in colaboration with Professor W. W. Johnson, appeared their Treatise on Differential Calculus,—a meritorious work.

Alexander Macfarlane, M. A., D. Sc., LL. D., delivered six lectures entitled "British Mathematicians of the Nineteenth Century," in the Chemical Laboratory of Lehigh University, beginning April 12, 1901. The life and work of the following mathematicians were presented in the order and at the time named below: George Peacock (1791-1858), April 12th, 11:30 a. m.; Augustus DeMorgan (1806-1871), April 13th, 11:30 a. m.; Sir William Rowan Hamilton (1805-1865), April 16th, 4:00 p. m., George Boole (1815-1864), April 19th, 11:30 a. m.; Arthur Cayley (1821-1895), April 20th, 11:30 a. m.; William Kingdon Clifford (1845-1879), April 23rd, 4:00 p. m.

BOOKS AND PERIODICALS.

Teachers' Manual of School Arithmetic. By J. M. Colaw, A. M., Assistant Editor of THE AMERICAN MATHEMATICAL MONTHLY, Montery, Va., and John K. Ellwood, A. M., Principal of the Colfax School, Pittsburg, Pa., author of Table Book and Test Problems in Mathematics. 8vo. cloth, 301 pages. Richmond, Va.: B. F. Johnson Publishing Co.

The first fourteen pages of this Manual is devoted to a brief history of Arithmetic, which is of great interest and value. Pages 19 to 51 are devoted to the teaching of Arithmetic, in which valuable suggestions to teachers are set forth and the best methods of teaching the subject are presented. The remaining part of the book contains excellent solutions of the problems in the authors' School Arithmetic and some solutions of special problems. B. F. F.

On Continuity and Irrational Numbers and the Nature and Meaning of Numbers. By Richard Dedekind. Translated by Wooster Woodruff Beman, Professor of Mathematics in the University of Michigan. 8vo, red cloth, 115 pages. Price, 75 cents. Chicago: The Open Court Publishing Co.

The lovers of Mathematics who are only familiar with the English language are again placed under great obligation to The Open Court Publishing Company and to Professor Beman for bringing out this translation of Dedekind's Memoirs on Continuity and Irrational Numbers, and the Nature and Meaning of Numbers. Every teacher of Mathematics who has not read these Memoirs in the original ought to buy this little book and read it carefully several times. B. F. F

Plane and Solid Analytical Geometry. An Elementary Text-Book. By Charles H. Ashton, A. M., Instructor in Mathematics in Harvard University. 8vo. cloth and leather back, xiii+266 pages. Price, $1.00. New York: Charles Scribner's Sons.

This book is intended for classroom use and is designed to meet the needs of classes having from sixty to seventy hours to devote to the subject. The conics are treated from their ratio definitions,—a method that commends itself to most teachers. Numerous well selected problems are inserted at the end of nearly every chapter. Two chapters are devoted to loci, and Poles and Polars are treated by the aid of harmonic divisions. The treatment of Analytic Geometry of Space is commendable, and Quadratic Surfaces here receive brief consideration. B. F. F.

A Manual of Laboratory Physics. By H. M. Tory, M. A., Lecturer in Mathematics, McGill University, Montreal; late Demonstrator of Physics, McDonald Physics Building, McGill University, and F. H. Pitcher, M. Sc., A. M. I. E. E., late Demonstrator of Physics, McDonald Physics Building, McGill University, Montreal. Crown 8vo., cloth, ix+288 pages. Price, $2.00. New York: John Wiley & Sons.

This book presents an elementary Laboratory Course in Sound, Light, Heat, Magnetism, and Electricity. Each experiment is carefully described. Under "Apparatus Required" is given an exact statement of the apparatus necessary for the particular experiment. Under "Theory of Experiment" is set forth the theory involved in the special experiment under consideration. Under "Practical Directions" are given such directions as a director would give if standing beside the student. In addition, a tabulated example of the observations and results are added as a guide to the student. A corresponding "Blank" to be filled by the student is added, and thus is provided a permanent record of his work. We consider this Manual meritorious in every particular, B. F. F.

Annals of Mathematics. Published under the auspices of Harvard University. Price, $2.00 per year in advance. Second Series, Vol. II., No. 3.

The April number contains the following articles: "Sufficient Conditions in the Calculus of Variations," by Professor W. F. Osgood; Lagrange's "Equation in the Calculus of Variations, and The Extension of a Theorem," by J. K. Whittemore; "On Some Points in the Theory of the Hypergeometric Function, Expressed as a Double Circuit Integral," by R. M. Hathaway; "A Theory of Continued Fractions," by Dr. Derrick N. Lehmer; "Note on the Dual of a Focal Property of the Inscribed Ellipse," by Professor R. E. Allardice; "A Simplified Solution of the Cubic," by Dr. Emory McClintock.
B. F. F.

Le Mathematiche, Vol. I., No. 2.

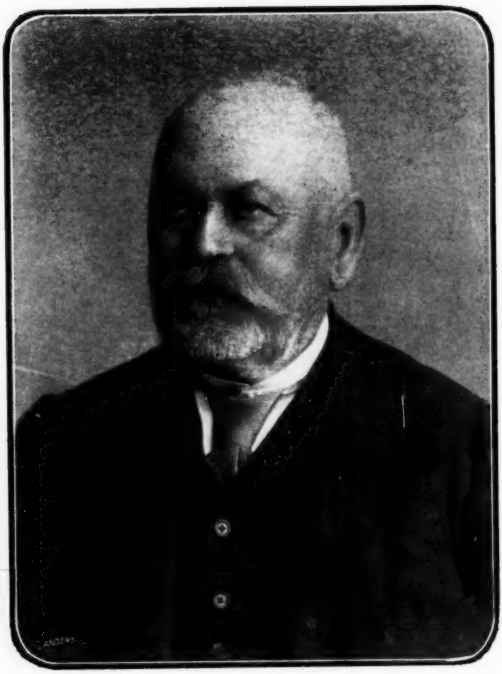
All American contributions to this journal should be sent to Dr. George Bruce Halsted, University of Texas, Austin, Texas. B. F. F.

ERRATA.

Page 26, line 13, for "Wolfgan" read Wolfgang.
Page 27, line 21, for "Emporer" read Emperor.
Page 28, line 14, for "*Varein*" read *Verein*.
Page 28, line 2 from bottom, for "Derichlet" read Dirichlet.
Page 29, line 20, for "elleptic" read elliptic.
Page 29, line 26, for "corporium" read corporum.
Page 30, line 13, for "michanique" read mecanique.
Page 31, line 4, for "imaginery" read imaginary.
Page 56, line 10, for "Homogenious" read Homogeneous.
Page 56, line 11, for "Quarternary" read Quaternary.

The above errors were pointed out to us by W. J. Greenstreet, Editor of *The Mathematical Gazette.* We shall appreciate the kindness if others will follow Professor Greenstreet's example, as we wish to rectify all errors. Also the following are noticed:

Page 28, line 2 from bottom, for "Vorlesunger" read Vorlesungen.
Problem 130, page 54, next to last line, after "$x^n-y^n=a$," insert $x-y=b$.



FRANZ SCHMIDT.

THE AMERICAN MATHEMATICAL MONTHLY.

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VOL. VIII. MAY, 1901. No. 5.

BIOGRAPHY.

FRANZ SCHMIDT.

BY DR. GEORGE BRUCE HALSTED.

On the fifteenth of December, 1902, will be fitly celebrated in Hungary the centenary of the birth of Bolyai János, but the man through whom the world come to possession of this truest genius will not be present.

On the seventh of March, 1901, at Budapest, in his seventy-fifth year, died the architect Franz Schmidt, whose name will ever be justly connected with that of John Bolyai.

Until my investigations in 1896 at Budapest, Kolozsvár and Maros-Vásárhely, what was known of Bolyai came wholly from a brief article by Franz Schmidt, then living at Temesvár, entitled "Aus dem Leben zweier ungarischer Mathematiker, Johann und Wolfgang Bolyai von Bolya," Grunert's Archiv, Bd. 48, 1868, page 217. This was translated into Italian by Angelo Forti under the title: "Intorno alla vita ed agli scritti di Wolfgang e Giovanni Bolyai di Bolya matematici ungheresi," and published in the Bulletino di Bibliografia e di storia delle scienze matematiche et fisiche. T. I. Rom, 1868.

But Schmidt had written the article at the prompting of Hoüel, who translated it into French and published it first in the Mémoires de la Société des sciences physiques et naturelles de Bordeaux, Tome V, Bordeaux 1867. He afterward prefixed it to his translation of John Bolyai's Science Absolute of Space in 1868.

In a "Note du Traducteur" Hoüel says: "The name of Bolyai has become inseparable from these profound discoveries. It was therefore a duty for European science to draw this name from an unjust oblivion.

The author of the Notice of which we publish the translation, M. Fr. Schmidt, has consecrated himself to this work of reparation with an indefatigable devotion not discouraged either by the slowness of communication between the different parts of the Hungarian empire, or by the difficulty of procuring, even in the native state of Bolyai, the necessary details about the man who has rendered it celebrated.

It is also to the learned architect of Temesvár that I owe the possession of the excessively rare work of John Bolyai, which I have reproduced after the Notice of its author, certain that it cannot fail to excite the interest of all those who truly love science.

By this publication, la Société des Sciences physiques et naturelles continues the work which it commenced by inserting in its preceding volume the researches of Lobachevski on the same subject. May this homage rendered to the Hungarian geometer decide his compatriots to produce from his papers, deposited in the College of Maros-Vásárhely, the remarkable works which they must contain, and of which we know as yet only the titles."

It was from this same copy of the Latin text sent from Temesvár to Hoüel by Schmidt, sent to Schmidt by Professor Samuel Szabó, that in 1868 G. Battaglini made the Italian version published by him in his Giornale di Matematiche.

Schmidt begins his paper as follows [I translate from a copy he presented me while I was with him at Budapest in his peaceful home on the banks of the beautiful blue Danube, at Rudolf-rakpart 8]: "The biographical sketch that I here trace is still very incomplete; but as there seems little likelihood of the appearance soon of a detailed life-history, I publish what I have been able to learn from information printed, oral or manuscript, in the hope that any who are in position to add to or make more precise these data, may be induced soon to complete what is here given." A quarter of a century passed away before the world knew anything further of John Bolyai.

In June, 1891, appeared my translation: "THE SCIENCE ABSOLUTE OF SPACE *Independent of the Truth or Falsity of Euclid's Axiom XI (which can never be decided a priori)*. By John Bolyai. In the introduction to the first edition of this I say: "Beyond the *Appendix*, whose translation into English is here given for the first time, John Bolyai published nothing; and the thousand pages of manuscript which he left have never been read by a competent mathematician. They are in the library of the Reformed College of Maros-Vásárhely. What discoveries might lie hidden in his papers!" So I formed the project of making the journey from Austin to Maros-Vásárhely, and entered into correspondence with Professar Koncz, a pupil of John Bolyai's father, who succeeded to his professorial chair and residence.

Professor Koncz wrote me in Latin, and finally sent me precious documents which I planned to incorporate in a more extensive Life of Bolyai. On